DISTRIBUTIVE LATTICE ORDERINGS
AND PRIESTLEY DUALITY

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ABSTRACT. The ordering relation of a bounded distributive lattice L is a (distributive) $(0, 1)$-sublattice of $L \times L$. This construction gives rise to a functor Φ from the category of bounded distributive lattices to itself. We examine the interaction of Φ with Priestley duality and characterise those bounded distributive lattices L such that there is K with $\Phi(K) \cong L$.[1]

1. SOME CONVENTIONS AND DEFINITIONS

For any poset P we say that $A \subseteq P$ is a *lower set* or *down-set* if $a \in A$, $x \in P, x \leq a$ imply $x \in A$. The dual notion is that of an *up-set*.

We assume all lattices to be distributive and bounded by 0,1 such that $0 \neq 1$. A nonempty down-set I of a bounded distributive lattice L is said to be an *ideal* if $a, b \in I$ implies $a \vee b \in I$. An up-set with the dual property is called a *filter*. Moreover, I is a *prime ideal* if $I \neq L$ and if $a, b \in L \setminus I$ implies $a \wedge b \in L \setminus I$. Note that a down-set of L is a prime ideal if and only if its complement is a filter.

Let L be a bounded distributive lattice. Then by $\mathcal{I}_p(L)$ we denote the set of all prime ideals of L. Suppose that $a \in L$, then we define

$$X_a^L = \{I \in \mathcal{I}_p(L) : a \notin I\}.$$

When no confusion arises, we omit the superscript and write X_a.

2. Priestley duality

In [5], Priestley proved that the category $\mathcal{D}_{01}$ of bounded distributive lattices with $(0,1)$-preserving lattice homomorphisms and the category $\mathcal{P}$ of compact totally order-disconnected spaces (henceforth referred to as *Priestley spaces*) with order-preserving continuous maps are dually equivalent. A compact *totally order-disconnected space* $(X; \tau, \leq)$ is a poset $(X; \leq)$ endowed with a compact topology τ such that, for $x, y \in X$, whenever $x \not\geq y$, then there exists a clopen decreasing set U such that $x \in U$ and $y \notin U$. We usually refer to a Priestley space by its ground set only when there is no ambiguity about the topology and the ordering relation being used. In the following we briefly describe the pair of contravariant functors connecting $\mathcal{D}_{01}$ and $\mathcal{P}$.

The functor $\mathcal{X} : \mathcal{D}_{01} \rightarrow \mathcal{P}$ assigns to each object L of $\mathcal{D}_{01}$ a Priestley space $(\mathcal{I}_p(L); \tau(L), \subseteq)$, where $\mathcal{I}_p(L)$ is the set of all prime ideals of L and the topology $\tau(L)$ is given by the following subbasis of $\mathcal{I}_p(L)$:

$$\{X_a : a \in L\} \cup \{\mathcal{I}_p(L) \setminus X_a : a \in L\}.$$

As lined out in [2], this topology is compact and totally order-disconnected; moreover it turns out that the collection of clopen down-sets consists exactly of the X_a (and the collection of clopen up-sets are their complements).

For a $(0,1)$-preserving lattice homomorphism $f : L \rightarrow K$ we define $\mathcal{X}(f) : \mathcal{X}(K) \rightarrow \mathcal{X}(L)$ by $\mathcal{X}(f)(I) = f^{-1}(I)$ for all $I \in \mathcal{X}(K) = \mathcal{I}_p(K)$, using the fact that preimages of prime ideals are prime ideals.

The functor $\mathcal{E} : \mathcal{P} \rightarrow \mathcal{D}$ assigns to each Priestley space set $\mathcal{E}(X)$ of all clopen down-sets of X ordered by set inclusion, which gives rise to a bounded distributive lattice.

On the level of morphisms, i.e. order-preserving continuous maps, $\mathcal{E}$ again works with preimages.

An excellent introduction to Priestley duality can be found in [2].

3. The functor Φ

Let $(P, \leq_P)$ be a nonempty poset. The cartesian product $P \times P$ of the underlying set P can be endowed with the coordinatewise

ordering, i.e. in $P \times P$ we have

$$(p, q) \leq_{P \times P} (p', q') \text{ if and only if } (p \leq_P p' \text{ and } q \leq_P q').$$

Since the ordering relation $\leq_P$ of P is a subset of $P \times P$ it inherits the ordering described above such that it can be regarded as a poset in its own right. We denote this poset constructed using $\leq_P$ as underlying set by $\Phi(P)$. Note that trivially by definition $\Phi(P)$ is a subposet of $P \times P$ and we have $\Phi(P) = P \times P$ if and only if P is a singleton.

If P, Q are posets and $f : P \to Q$ is an order-preserving function, it is easily seen that the restriction of

$$(f \times f) : P \times P \to Q \times Q \text{ defined by } (p_1, p_2) \mapsto (f(p_1), f(p_2))$$

to $\Phi(P)$ gives rise to an order-preserving function

$$\Phi(f) : \Phi(P) \to \Phi(Q).$$

It is easy to verify that with this construction we can make Φ into a functor from the category of posets with order-preserving functions to itself.

Another easy calculation shows that if L is a lattice then so is $\Phi(L)$. Operations are componentwise; indeed $\Phi(L)$ is a sublattice of the lattice $L \times L$. For $L \in \mathcal{D}_{01}$ it turns out that $\Phi(L)$ is a $(0, 1)$-sublattice of $L \times L$ and therefore $\Phi(L) \in \mathcal{D}_{01}$.

Perhaps not surprisingly, given a $(0, 1)$-lattice homomorphism $f : L \to K$ between bounded (not necessarily distributive) lattices, the map $\Phi(f) : \Phi(L) \to \Phi(K)$ is a lattice $(0, 1)$-homomorphism as well. Routine verification shows that $L \mapsto \Phi(L)$ and $f \mapsto \Phi(f)$ gives rise to a functor $\Phi : \mathcal{D}_{01} \to \mathcal{D}_{01}$. This is what we want to have a closer look at in the following. In section 4 we calculate $\mathcal{I}_p(\Phi(L))$ in terms of $\mathcal{I}_p(L)$ and in section 5 we look at the interaction of Φ with Priestley duality and characterise those bounded distributive lattices L such that there is K with $\Phi(K) \cong L$.

A similar and in some way more general construction was studied by J.D. Farley in [3].

4. Calculating $\mathcal{I}_p(\Phi(L))$

In this section we express the collection of prime ideals of $\Phi(L)$ in terms of $\mathcal{I}_p(L)$.

Lemma 4.1. *If S is a ideal of $\Phi(L)$ then*

$$S = (pr_1(S) \times pr_2(S)) \cap \Phi(L)$$

where $pr_j : L \times L \to L$ is defined by $(l_1, l_2) \mapsto l_j$ for $j = 1, 2$.

Proof. Certainly $S \subseteq (pr_1(S) \times pr_2(S)) \cap \Phi(L)$. On the other hand suppose that $(a, b) \in (pr_1(S) \times pr_2(S)) \cap \Phi(L)$. So there is $b_a, a_b \in L$ such that $(a, b_a), (a_b, b) \in S$. Therefore $(a \vee a_b, b \vee b_a) \in S$ which entails $(a, b) \in S$, since S is an ideal. $\qquad\square$

For notational convenience, let S_i denote $pr_i(S)$ for $i = 1, 2$. Note that $S_1 = \{a \in L : (\exists b \in L) : (a, b) \in S\}$. For S_2, a similar statement holds.

Lemma 4.2. *If S is a prime ideal of $\Phi(L)$ then $S_1 \in \mathcal{I}_p(L)$ and $S_2 \in \mathcal{I}_p(L) \cup \{L\}$.*

Proof. First note that $S_1 \neq L$: for if we had $1 \in S_1$, then there would be $b \in L$ such that $(1, b) \in S \subseteq \Phi(L)$, so $b = 1$. But if S contains $(1, 1)$ then we have $S = \Phi(L)$.

Moreover it is fairly easy to see that S_1, S_2 are ideals. Now suppose that $c, d \notin S_1$ but $c \wedge d \in S_1$. So there is $b \in L$ such that $(c \wedge d, b) \in S$. Now $S \subseteq \Phi(L)$ entails $c \wedge d \leq b$. Since S is a down-set of $\Phi(L)$, certainly $(c \wedge d, c \wedge d) \in S$. Moreover we have $(c, c), (d, d) \notin S$ (because $c, d \notin S_1$), so $(c, c) \wedge (d, d) = (c \wedge d, c \wedge d) \in S$ which contradicts S being prime. With a similar argument we show that S_2 has the "prime property" - although it is possible that $S_2 = L$. $\qquad\square$

Lemma 4.3. *If S is a prime ideal of $\Phi(L)$ then $S_2 \in \{S_1, L\}$.*

Proof. First we show $S_2 \supseteq S_1$. Let $a \in S_1$, so by definition of S_1 there exists $b \in L$ such that $(a, b) \in S \subseteq \Phi(L)$. By construction of $\Phi(P)$ this implies $a \leq b$. Note that S is a down-set of $\Phi(P)$ and $(a, a) \in \Phi(P)$ by definition of Φ. Moreover, $(a, a) \leq (a, b)$ in $\Phi(P)$ since $\Phi(P)$ is ordered coordinatewise. Because S is a down-set, one obtains $(a, a) \in S$ and therefore $a \in S_2$ by definition of S_2.

Now suppose that S_2 is a proper superset of S_1. We want to show that $S_2 = L$. Suppose $1 \notin S_2$. Take $y \in S_2 \setminus S_1$. There is $a \in L$ such that $(a, y) \in S$ (in particular $a \leq y$). So $(y, y) \notin S$ and $(a, 1) \notin S$ (because $1 \notin S_2$), but $(a, 1) \wedge (y, y) = (a, y) \in S$, contradicting S being prime. $\qquad\square$

Corollary 4.4. *For the lattice L we have,*

$$\mathcal{I}_p(\Phi(L)) = \{(I \times I) \cap \Phi(L); I \in \mathcal{I}_p(L)\} \cup \{(I \times L) \cap \Phi(L); I \in \mathcal{I}_p(L)\}.$$

Proof. It is straightforward to check that $(I \times I) \cap \Phi(L)$ and $(I \times L) \cap \Phi(L)$ are prime ideals of $\Phi(L)$ whenever I is a prime ideal of L.

On the other hand, suppose that $S \in \mathcal{I}_p(\Phi(L))$. By Lemma 4.1 the prime ideal S can be written as $(S_1 \times S_2) \cap \Phi(L)$. From Lemma 4.2 we get that $I := S_1$ is prime and finally Lemma 4.3 implies that S is either $(I \times I) \cap \Phi(L)$ or $(I \times L) \cap \Phi(L)$. □

5. When is L isomorphic to $\Phi(K)$ for some K?

The following question arises naturally: When is a bounded distributive lattice L isomorphic to $\Phi(K)$ for some bounded distributive lattice K? One special Priestley space will be the key here. Denote by $\underline{2}$ the ordinal $2 = \{0, 1\}$ with its standard ordering and the discrete topology.

With the aid of Priestley duality we are able to give an answer to that question. Let the pair of functors be denoted by $\mathcal{X} : \mathcal{D}_{01} \to \mathcal{P}$ and $\mathcal{E} : \mathcal{P} \to \mathcal{D}_{01}$ where $\mathcal{P}$ denotes the category of Priestley spaces with order-preserving continuous functions.

Lemma 5.1. *Let X be a Priestley space. Then*
  (1) $\Phi(\mathcal{E}(X)) \cong \mathcal{E}(X \times \underline{2})$ *in $\mathcal{D}_{01}$ and dually*
  (2) $\mathcal{X}(\Phi(L)) \cong \mathcal{X}(L) \times \underline{2}$ *in $\mathcal{P}$.*

Proof. For the first statement, consider the function

$$\varphi : \Phi(\mathcal{E}(X)) \to \mathcal{E}(X \times \underline{2}) \text{ defined by } (d, e) \mapsto d \times \{1\} \cup e \times \{0\}$$

for clopen down-sets d, e of X, and also

$$\psi : \mathcal{E}(X \times \underline{2}) \to \Phi(\mathcal{E}(X)) \text{ defined by } c \mapsto (c_1, c_0)$$

for each clopen down-set c of X where $c_i := \{x \in X : (x, i) \in c\}$ for $i = 0, 1$. We claim that φ and ψ are order-preserving inverses of each other and therefore provide an order (and lattice) isomorphism between $\Phi(\mathcal{E}(X)) \cong \mathcal{E}(X \times \underline{2})$. First note that for $(d, e) \in \Phi(\mathcal{E}(X))$ we have $d \subseteq e$ and therefore $\varphi((d, e)) = (d \times \{1\}) \cup (e \times \{0\})$ is a clopen down-set in $X \times \underline{2}$. On the other hand, if c is a clopen down-set of $X \times \underline{2}$ then $c_0 \supseteq c_1$, and clearly c_0, c_1 are clopen down-sets of X, so $(c_1, c_0) \in \Phi(\mathcal{E}(X))$. It is straightforward to check that φ

and ψ are both order-preserving, so it remains to show that they are inverses of each other. Note that $\varphi(d, e)_1 = d$ and $\varphi(d, e)_0 = e$ for clopen down-sets d, e of X. So $\psi(\varphi(d, e)) = (d, e)$. Moreover for any clopen down-set c of $X \times \underline{2}$ we have $c = (c_1 \times \{1\}) \cup (c_0 \times \{0\})$, so $\varphi(\psi(c)) = c$.

As for the second statement, let $X := \mathcal{X}(L)$. If we apply the functor $\mathcal{X}$ to statement 1, we get $\mathcal{X}(\Phi(\mathcal{E}(X))) \cong \mathcal{X}(\mathcal{E}(X \times \underline{2}))$. So we get with that and Priestley duality:

$$\mathcal{X}(\Phi(L)) \cong \mathcal{X}(\Phi(\mathcal{E}(X))) \cong \mathcal{X}(\mathcal{E}(X \times \underline{2})) \cong X \times \underline{2}$$

which proves statement 2. $\hfill\square$

Theorem 5.2. *For $L \in \mathcal{D}_{01}$ the following statements are equivalent:*

(1) *L is isomorphic to $\Phi(K)$ for some bounded distributive lattice K*

(2) *The Priestley space $\mathcal{X}(L)$ is order-homeomorphic to $Y \times \underline{2}$ for some Priestley space Y.*

Proof. Let $L \cong \Phi(K)$. Then by Lemma 5.1, statement 2, we get

$$\mathcal{X}(L) \cong \mathcal{X}(\Phi(K)) \cong \mathcal{X}(K) \times \underline{2}.$$

So, taking $Y := \mathcal{X}(K)$ we are done.

For the other directition, suppose $\mathcal{X}(L) \cong Y \times \underline{2}$. By Lemma 5.1, statement 1, we get

$$L \cong \mathcal{E}(\mathcal{X}(L)) \cong \mathcal{E}(Y \times \underline{2}) \cong \Phi(\mathcal{E}(Y)).$$

So, taking $K := \mathcal{E}(Y)$ we are done. $\hfill\square$

6. Fixed points of Φ

Another natural question arising in the context of the functor Φ is finding fixed points of Φ, that is, distributive $(0, 1)$-lattices L with the property that $\Phi(L) \cong L$.

With Theorem 5.2 we can say

> A lattice L is isomorphic to $\Phi(L)$ if and only if for the Priestley space Y of L has the property that $Y \cong Y \times \underline{2}$ (meaning there is an order-preserving homeomorphism from Y to $Y \times \underline{2}$).

The search for Priestley spaces Y with the property $Y \cong Y \times \underline{2}$ gives rise to an example of a fixed point of Φ.

Example 6.1. *For the Priestley space $Y = \underline{2}^\omega$ (endowed with the product topology and the coordinatewise ordering) we have $Y \cong Y \times \underline{2}$.*

The following map provides an order-preserving bijection from Y to $Y \times \underline{2}$:

$$y \mapsto (\mathrm{ls}(y), y(0)),$$

where ls denotes the left shift $\mathrm{ls} : \underline{2}^\omega \to \underline{2}^\omega$ given by $\mathrm{ls}(y)(n) = y(n+1)$ for all $n \in \omega$ and $y \in \underline{2}^\omega$. It is easy to see that the product topology on $\underline{2}^\omega$ coincides with the interval topology on the poset $\underline{2}^\omega$ which is true for $\underline{2}^\omega \times \underline{2}$ as well. Recall that the *interval topology* on any poset P is the topology generated by

$$\{P \setminus [x, y] : x, y \in P \text{ and } x \leq y\}.$$

Recall that $[x, y] = \{z \in P : x \leq z \leq y\}$. Note that any order-isomorphism between posets is a homeomorphism between the ground sets endowed with the interval topology. So the order-isomorphism from above is a homeomorphism as well, which proves that Y and $Y \times \underline{2}$ are homeomorphic in $\mathcal{P}$.

Applying Priestley duality to this example implies that for the lattice $L = \mathcal{E}(Y)$ we have $\Phi(L) \cong L$. The object L is (isomorphic to) the free distributive $(0, 1)$-lattice generated by countably many points.

It is unclear how to characterise those Priestley spaces Y with $Y \cong Y \times \underline{2}$. The functor Φ and its fixed points gives rise to more questions.

Question 6.2. *Does $\underline{2}^\omega$ embed into every Priestley space Y having the property that $Y \cong Y \times \underline{2}$? If not, is there a countable such Priestley space Y?*

Of course, the functor Φ can be studied in the more general settings of posets (even of preordered sets). Note that if P and Q are posets which are fixed points of Φ, then so is their disjoint union. So it is more rewarding to consider connected posets only. Recall that a poset $(P, \leq)$ is connected if

$$tr(\leq \cup (\leq)^{-1})) = P \times P$$

where $(\leq)^{-1} = \{(y, x) : x \leq y\}$ and tr denotes the transitive closure.

Question 6.3. *Is there a connected poset P with more than one point such that P is not a lattice and $\Phi(P) \cong P$?*

There are several "cardinal functions" in the category of posets. Let us just mention the order dimension and the width. The *width* is the supremum of all cardinalities of anti-chains of a poset $(P, \leq)$ where an anti-chain is a subset $A \subseteq P$ such that $x \neq y \in A$ implies $x \nleq y$ and $y \nleq x$. Moreover recall that any ordering relation equals the intersection of all total ordering relations containing it. (In a *total* ordering relation we have $x \leq y$ or $y \leq x$ for all x, y in the ground set.) The *order dimension* of a poset $(P, \leq)$ is the minimal cardinality κ such that there is a collection $\mathcal{S}$ of total ordering relations such that the intersection of $\mathcal{S}$ equals the given ordering relation and $\mathrm{card}(\mathcal{S}) = \kappa$.

Natural questions arise when looking at those functions' interaction with Φ, especially in the case of finite posets. One example would be:

Question 6.4. *If P is a finite poset, how does its order dimension compare to that of $\Phi(P)$?*

7. Acknowledgement

References

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